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Relativistic Positioning Systems and Gravitational Perturbations

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Abstract. In order to deliver a high accuracy relativistic positioning system, several gravitational perturbations need to be taken into account. We therefore consider a system of satellites, such as the Galileo system, in a space-time described by a background Schwarzschild metric and small gravitational perturbations due to the Earth's rotation, multipoles and tides, and the gravity of the Moon, the Sun, and planets. We present the status of this work currently carried out in the ESA Slovenian PECS project *Relativistic Global Navigation System*, give the explicit expressions for the perturbed metric, and briefly outline further steps.

1 Introduction

Current Global Navigation Satellites Systems (GNSS), such as the Global Positioning System and the European Galileo system, are based on Newtonian concept of absolute time and space. The signals from four satellites are needed for a receiver to determine its position and time via the time difference between the emission and the reception of the signal. However, due to the inertial reference frames and curvature of the space-time in the vicinity of Earth, space and time cannot be considered as absolute. In fact, general relativistic effects are far from being negligible [2]. Current GNSS deal with this problem by adding general relativistic corrections to the level of the accuracy desired. An alternative, and more consistent, approach is to abandon the concept of absolute space and time and describe a GNSS directly in general relativity, i.e. to define a Relativistic Positioning System (RPS) with the so-called emission coordinates [5, 16, 3, 4, 17]. A user of an RPS receives, at a given moment, four signals from four different satellites. It is able to determine the proper time τ of each satellite at the moment of emission of these signals. Then these four proper times $(\tau_1, \tau_2, \tau_3, \tau_4)$ constitute its emission coordinates. By receiving them at subsequent times, the receiver therefore knows its trajectory in the emission coordinates.

There are several advantages of an RPS. Firstly, the emission coordinates are covariant quantities; they are

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independent of the observer (although dependent on the set of satellites chosen and their trajectories). Moreover, if each satellite broadcasts its own and also receives proper times of other satellites, the system of satellites is autonomous and constitutes a primary reference system, with no need to define a terrestrial reference frame. Therefore tracking of satellites with ground stations is necessary only to link an RPS to a terrestrial frame, although this link can also be obtained by placing several receivers at the known terrestrial positions. There is no need to synchronize satellite clocks to a time-scale realized on the ground (like it is done today with GPS time, which is a realization of the TT Terrestrial Time). No relativistic corrections are necessary, as relativity is already included in the definition of the positioning system.

To demonstrate feasibility, stability and accuracy of an RPS, two ESA Ariadna projects were carried out in 2010 and 2011 [6, 7, 8]. In these projects, the authors modelled an RPS in the idealized case of Schwarzschild geometry and found that relativistic description of a satellite constellation with inter-satellite links provides numerically accurate, stable and autonomous system. They called such a system the Autonomous Basis of Coordinates (ABC). By communicating their proper times solely, two satellites can determine their orbits (i.e., their constants of motion). Any additional satellite would serve to increase the system's accuracy.

Outline of the PECS Relativistic GNSS project and its goals

Continuation of this work is presently being carried out in the ESA Slovenian PECS project *Relativistic Global Navigation System* (2011-2014). In this project, we aim to demonstrate that an RPS and the ABC concept are highly accurate and stable also if the space-time is not purely spherically symmetric, but contains small gravitational perturbations due to the Earth's multipoles, tides and rotation and gravitational influences of the Moon, the Sun, Jupiter and Venus. In this work we use the same approach that was used in two Araidna projects (description of satellites' orbit in the ABC, emission coordinates, inter-satellite communication and recovery of their orbits) and add to a Hamiltonian gravitational perturbations.

In brief: we start modelling of an RPS by writing the perturbed Hamiltonian, corresponding equations of motion and calculating orbits of satellites. Then we define the system based on the ABC with four satellites, simulate their inter-satellite communication and use the emission/reception of their proper times to determine their orbital parameters. Our work is divided in the following steps:

1 Add first order gravitational perturbations to the Schwarzschild metric: find perturbation coefficients describing all known gravitational perturbations, i.e., due to Earth's multipoles, tides and rotation; gravity of the Moon, the Sun, and planets (Venus and Jupiter).

2 Solve the perturbed geodesic equations: use Hamiltonian formalism and perturbation theory to obtain time evolution of zeroth order constants of motion. Simulate satellites' orbits.

3 Find accurate constants of motion: use solely inter-satellite links over many orbital periods to determine satellites' orbital parameters and study the stability and possible degeneracies between them.

4 Refine values of gravitational perturbation coefficients: use additional satellites and residual errors between orbit prediction and orbit determination through inter-satellite communication to improve the accuracy of position determination and to probe the space-time, i.e., measure perturbations. Discuss possible scientific applications.

Before starting detailed calculations it is useful to get an estimation of the order of magnitude of various perturbations on a satellite orbit (as in e.g. [13, Fig. 3.1.]). At GNSS altitudes of about 20.000 km above Earth, the most important gravitational perturbations are due to Earth's multipoles, followed by the gravitational field of the Moon and the Sun. Several orders of magnitude smaller are perturbations due to Solar radiation pressure and the Earth's albedo (not considered in our project) and due to Earth's tides. About one order of magnitude smaller are relativistic effects and the gravitational influence of Jupiter and Venus. Relativistic effects due to Earth's rotation are about an order of magnitude smaller again.

Because our project is still in progress, we will present here only the results of the first part of the project: we will give the expressions for the perturbed metric around Earth.

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2 Metric perturbations in the Schwarzschild background

We describe the spherically symmetric and time independent background with the Schwarzschild metric $g^{(0)}_{\mu\nu}$ and denote metric perturbations with $h_{\mu\nu}$. Because gravitational perturbations are several orders of magnitude smaller than the Earth's gravitational GM term ($h_{\mu\nu} \ll g^{(0)}_{\mu\nu}$), we use linear perturbation theory and write the perturbed metric as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \,. \tag{1}$$

Because we are interested in the space-time outside Earth, the perturbed metric must satisfy the Einstein equation for vacuum:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2}$$

where

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \delta R_{\mu\nu} \tag{3}$$

is the Ricci tensor (symbol $^{(0)}$ denotes unperturbed quantities and δ perturbations). The Einstein equation becomes:

$$\begin{aligned} &h_{\alpha}{}^{\alpha}{}_{;\mu\nu} - h_{\mu}{}^{\alpha}{}_{;\nu\alpha} - h_{\nu}{}^{\alpha}{}_{;\mu\alpha} + h_{\mu\nu;}{}^{\alpha}{}_{\alpha} \\ &+ g^{(0)}_{\mu\nu}(h_{\alpha}{}^{\lambda}{}_{;}{}^{\alpha}{}_{\lambda} - h_{\lambda}{}^{\lambda}{}_{;\alpha}{}^{\alpha}) - h_{\mu\nu}R^{(0)} \\ &+ g^{(0)}_{\mu\nu} h_{\lambda\sigma} R^{(0)\lambda\sigma} = 0 \end{aligned} \tag{4}$$

where a semi-colon (;) denotes covariant derivative, calculated with respect to the unperturbed metric $g^{(0)}_{\mu\nu}$.

To find solutions of these equations for vacuum we use the Regge-Wheeler-Zerilli (RWZ) framework [15, 18] . In the RWZ formalism, the metric perturbation $h_{\mu\nu}$ is expanded into a series of independent tensor harmonics, a tensor analog to spherical harmonic functions, labeled by indices n (degree), m (order), and parity: odd or even. By adopting the notation from [14], the general expansion of the metric perturbation $h_{\mu\nu}$ can be written as

$$h_{\mu\nu} = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} (h^{nm}_{\mu\nu})^{(o)} + (h^{nm}_{\mu\nu})^{(e)} \,, \tag{5}$$

where the expansion terms $(h^{nm}_{\mu\nu})^{(o)}$ and $(h^{nm}_{\mu\nu})^{(e)}$ are the odd-parity and the even-parity metric functions (or modes), respectively. We find it most convenient to work in the gauge from [15], where the odd parity metric functions are

$$(h^{nm}_{\mu\nu})^{(o)} = \left[\begin{array}{cc|cc} 0 & 0 & -h_0 \csc\theta\partial_\varphi & h_0 \sin\theta\partial_\theta \\ 0 & 0 & -h_1 \csc\theta\partial_\varphi & h_1 \sin\theta\partial_\theta \\ \hline \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{array}\right] Y^m_n \,, \tag{6}$$

and for even parity, the metric functions are

$$(h^{nm}_{\mu\nu})^{(e)} = \left[\begin{array}{cc|cc} H_0\chi & H_1 & 0 & 0 \\ \star & H_2\chi^{-1} & 0 & 0 \\ \hline 0 & 0 & r^2K & 0 \\ 0 & 0 & 0 & r^2K\sin^2\theta \end{array}\right] Y^m_n \,, \tag{7}$$

where $\star$ indicates the symmetric part of the tensor, $\chi = 1 - r_s/r$, and $r_s = 2GM/c^2$ is the Schwarzschild radius. Expressions h_i, H_i, and K depend on Schwarzschild coordinates (t, r) and indices (n, m), which are omitted in the expressions for clarity. It is shown in [15, 18] that in vacuum $H_0 = H_2$, therefore, both functions are marked with H.

After inserting (6) and (7) into (4), we obtain a number of partial differential equations. We find and write their solutions for time-independent and time-dependent perturbations, separately.

2.1 Time-independent metric perturbations

Even-parity contributions

In the case of time-independent perturbations of even parity $H_1 = 0$ [15]. It follows that the even metric mode (7) is diagonal:

$$(h^{nm}_{\mu\nu})^{(e)} = \mathrm{diag}(H\chi, H\chi^{-1}, r^2K, r^2K\sin^2\theta) Y^m_n \,. \tag{8}$$

Inserting it in (4) gives equations for H and K. We find the solution for H to be:

$$H(r) = A_{nm} \frac{P^{(0)}_n\left(\frac{r_s}{r}\right)}{r^n(r - r_s)} + B_{nm} \frac{r^{n+1} R^{(0)}_n\left(\frac{r_s}{r}\right)}{r - r_s} \,, \tag{9}$$

where the functions $P^{(0)}_n$ and $R^{(0)}_n$ are Gaussian hypergeometric functions $_2F_1$ [1] (for more details see[11]).

The solution for function K is determined by H (see [11]) and can be written as

$$K(r) = A_{nm} \frac{P^{(1)}_n\left(\frac{r_s}{r}\right)}{r^{n+1}} + B_{nm} r^n R^{(1)}_n\left(\frac{r_s}{r}\right) \,, \tag{10}$$

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where functions $P_n^{(1)}$ and $R_n^{(1)}$ can be expressed via $P_n^{(0)}$ and $R_n^{(0)}$ [11].

In our case of satellites in orbit around Earth, time-independent perturbations are due to the Earth's multipoles (for the time being, we neglect Earth's rotation and tides). In order to determine the constants A_{nm} and B_{nm} in (9) and (10), we compare equation (9) with its Newtonian counterpart, i.e., the gravitational potential Φ. For a non-rotating object of mass M the potential Φ can be expanded into a series of multipole contributions (eq. 3.61 in [12]):

$$\begin{aligned}\Phi = &\frac{GM}{r}\\ &+ \sum_{nm}(M^{\oplus}_{nm} r^{-n-1} + M^{\ominus}_{nm} r^{n}) Y^m_n\,,\end{aligned} \tag{11}$$

where $M^{\oplus}_{nm}$ and $M^{\ominus}_{nm}$ are time-independent spherical multipole momenta and notation $\sum_{nm} \equiv \sum_{n=2}^{\infty}\sum_{m=-n}^{n}$ is used. The first term in the sum describes the gravitational potential of the perturbing sources positioned within the radius r, while the second term corresponds to those outside r. Comparing (9) with (11), we notice the same behavior (i.e., the superposition of r^{+n} and r^{-n-1} functional dependence) in the perturbative part of (11) and it is evident that the coefficients A_{nm} and B_{nm} are related to the multipole momenta. The relation between both is found from the weak field approximation

$$\frac{c^2}{2}(1+g_{00}) \sim \Phi\,. \tag{12}$$

By inserting

$$\begin{aligned} g_{00} &= g_{00}^{(0)} + \sum_{nm} (h_{00}^{nm})^{(e)} \\ &= \chi\left(-1+\sum_{nm} H_{nm}\right)\end{aligned} \tag{13}$$

into the above relation together with the Newtonian potential (11), we find that in the weak field limit A_{nm} and B_{nm} are asymptotically related to Newtonian spherical multipole momenta $M^{\oplus}_{nm}$ and $M^{\ominus}_{nm}$ as

$$A_{nm} \sim \frac{2}{c^2} M^{\oplus}_{nm} \quad \text{and} \quad B_{nm} \sim \frac{2}{c^2} M^{\ominus}_{nm}\,. \tag{14}$$

Note that for finite c, $M^{\oplus}_{nm}$ and $M^{\ominus}_{nm}$ only approximate A_{nm} and B_{nm}.

Odd-parity contributions

In case of time-independent perturbations, the odd metric functions $(h^{nm}_{\mu\nu})^{(o)}$ in (6) have $h_1 = 0$ [15] and can be written with a single function h_0 as

$$\begin{aligned}(h^{nm}_{\mu\nu})^{(o)} = &- h_0 \csc\theta\, Y^m_{n\,,\varphi}(\delta_{0,\mu}\delta_{2,\nu} + \delta_{2,\mu}\delta_{0,\nu})\\ &+ h_0 \sin\theta\, Y^m_{n\,,\theta}(\delta_{0,\mu}\delta_{3,\nu} + \delta_{3,\mu}\delta_{0,\nu})\,.\end{aligned} \tag{15}$$

The solution for h_0 is:

$$h_0(r) = \alpha_{nm}\frac{P_n^{(2)}\left(\frac{r_s}{r}\right)}{r^n} + \beta_{nm} r^{n+1} R_n^{(2)}\left(\frac{r_s}{r}\right)\,, \tag{16}$$

where functions $P_n^{(2)}$ and $R_n^{(2)}$ are Gaussian hypergeometric functions ${}_2F_1$ (for more details see[11]).

To determine the constants α_{nm} and β_{nm} we note that off-diagonal terms in the metric tensor are associated with frame-dragging effects. In our case frame-dragging effects from 'external' objects (the Sun, the Moon, other planets) are negligible. Consequently, we set $\beta_{nm} = 0$.

We do take into account the frame-dragging due to the Earth's rotation. To determine α_{nm}, we notice that for $n = 1$ and $m = 0$ the corresponding h_0 matches the weak field and slow rotation approximation of the Kerr metric: if $r \gg r_s$ and Earth's angular parameter $a \ll 1$, then for $\alpha_{10} = a r_s\sqrt{4\pi/3}$ it follows

$$h_0(r) = a\frac{r_s}{r}\sqrt{\frac{4\pi}{3}}\,, \tag{17}$$

where we keep only the terms linear in a.

For higher multipoles ($n > 1$), it turns out that their dependence on a is not linear [10]. Therefore, the only multipole we include in the odd-parity metric function is the monopole, i.e., the one belonging to the linear (in a) part of the Kerr effect.

2.2 Time-dependent metric perturbations

Due to the Earth's rotation its multipoles vary periodically. Earth's solid and ocean tides introduce additional time dependency in its multipoles (additional variability with different frequency, phase and varying amplitude, depending on the position of the Moon and the Sun). In addition, the gravitational influence of other celestial bodies introduces time dependent perturbations to the space-time around Earth, because their relative positions with respect to Earth change with time. These

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perturbations can be expanded in a series of multipoles and treated with the same procedure as the Earth's multipoles.

We therefore consider time dependent metric perturbations for the case of perturbations oscillating slowly with angular velocities, which are smaller or of the same order of magnitude as the angular velocity of Earth. All angular velocities are defined with respect to the Schwarzschild time t.

Even-parity contributions

Even parity modes are connected to the Newtonian gravitational potential Φ, which in the case of time dependent multipoles can be written as:

$$\begin{aligned}\Phi = &\frac{GM}{r} \\ &+ \sum_{nm} (M^{\oplus}_{nm}(T) r^{-n-1} + M^{\ominus}_{nm}(T) r^{n}) Y^m_n \,,\end{aligned} \tag{18}$$

where $T = ct$, or alternatively in frequency domain:

$$\begin{aligned}\Phi = &\frac{GM}{r} + \sum_{nm} \int_{-\infty}^{\infty} dk\, e^{ikT} \times \\ &\left[\widetilde{M}^{\oplus}_{nm}(k) r^{-n-1} + \widetilde{M}^{\ominus}_{nm}(k) r^{n}\right] Y^m_n \,,\end{aligned} \tag{19}$$

where k is the wavenumber and $\widetilde{M}^{\oplus}_{nm}$, $\widetilde{M}^{\ominus}_{nm}$ are the Fourier transforms of time dependent multipoles:

$$\widetilde{M}^{u}_{nm}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dT\, e^{-ikT} M^{u}_{nm}(T) \,, \tag{20}$$

where $u = \oplus, \ominus$.

Each time dependent multipole generates a time-dependent even metric perturbation $(h^{nm}_{\mu\nu})^{(e)}$. Functions H, H_1, and K determining the modes can be expressed with their Fourier transforms as

$$\begin{aligned}(H(T), H_1(T), K(T)) = \\ \int_{-\infty}^{\infty} dk\, e^{ikT} (\widetilde{H}(k), \widetilde{H}_1(k), \widetilde{K}(k))\end{aligned} \tag{21}$$

Because in our case k is very small ($k \ll 1/r$), we solve differential equations for H, H_1, and K perturbativelly in k. We assume that $\widetilde{H}$, $\widetilde{H}_1$, and $\widetilde{K}$ are smooth functions of k and write them as a power series of dimensionless $\kappa = kr_s$:

$$(\widetilde{H}, \widetilde{H}_1, \widetilde{K}) \sim \sum_{i=0}^{\infty} \kappa^{2i} (\widetilde{H}^{(i)}, i\kappa \widetilde{H}^{(i)}_1, \widetilde{K}^{(i)}) \,, \tag{22}$$

The first terms in the expressions for H and K are already known: $\widetilde{H}^{(0)}$ is given in (9) and $\widetilde{K}^{(0)}$ in (10), where instead of (14) we use:

$$A_{nm} \sim \frac{2}{c^2} \widetilde{M}^{\oplus}_{nm} \quad \text{and} \quad B_{nm} \sim \frac{2}{c^2} \widetilde{M}^{\ominus}_{nm} \,. \tag{23}$$

For $\tilde{H}_1$ we find that

$$\begin{aligned}\widetilde{H}^{(0)}_1(r) = &A_{nm} \frac{r^{-n+1} P^{(3)}_n\left(\frac{r_s}{r}\right)}{r_s(r - r_s)} \\ &+ B_{nm} \frac{r^{n+2} R^{(3)}_n\left(\frac{r_s}{r}\right)}{r_s(r - r_s)} \,,\end{aligned} \tag{24}$$

where functions $P^{(3)}_n$ and $R^{(3)}_n$ are given as a series in $\frac{r_s}{r}$ [11].

Because k is very small for all time-dependent perturbations considered, we neglect all higher than leading terms in the expansion (22) and use the following approximations:

$$H(T, r) \approx \int_{-\infty}^{\infty} dk\, e^{ikT} \widetilde{H}^{(0)}(k, r) \,, \tag{25}$$

$$H_1(T, r) \approx \int_{-\infty}^{\infty} dk\, ikr_s e^{ikT} \widetilde{H}^{(0)}_1(k, r) \,, \tag{26}$$

$$K(T, r) \approx \int_{-\infty}^{\infty} dk\, e^{ikT} \widetilde{K}^{(0)}(k, r) \,. \tag{27}$$

A metric perturbation expressed with these functions is accurate up to the linear order in frequency. Since higher order perturbations naturally give rise to contributions with higher orders of frequencies, our approximation of a perturbation is consistently linear, i.e., it is linear in frequencies and in the order of perturbation.

Odd-parity contributions

For odd-parity solutions we use the same notation as in (21) and we find that the asymptotic behaviour of solutions $\tilde{h}_0$ and $\tilde{h}_1$ is not flat:

$$\tilde{h}_1(r) \asymp r \sin(kr + \phi) \tag{28}$$

$$\tilde{h}_0(r) \asymp r \cos(kr + \phi) \tag{29}$$

These solutions are therefore not relevant in our case.

3 Metric around Earth

Finally, we can write the metric perturbation $h_{\mu\nu}$, which in (5) was expressed as a series of normal modes

$(h^{nm}_{\mu\nu})^{(o)}$ and $(h^{nm}_{\mu\nu})^{(e)}$. Based on the positions of the sources of perturbations, these modes can be grouped into two terms:

$$h_{\mu\nu} = h^{\oplus}_{\mu\nu} + h^{\ominus}_{\mu\nu} . \tag{30}$$

The term $h^{\oplus}_{\mu\nu}$ represents the Earth's time dependent (exterior) multipoles and the frame-dragging effect of Earth. The former arise from the shape of the Earth, which changes with time due to rotation and tidal forces. For the latter, Kerr effect, there is no non-relativistic counterpart.

The term $h^{\ominus}_{\mu\nu}$ represents the time dependent (interior) multipoles of other celestial bodies. This term arises from the perturbative effect of other planets, the Moon, and the Sun, whose positions relative to Earth's change with time. Their frame-dragging effect is neglected.

To simplify expressions, we introduce the normalized complex multipoles ($u = \oplus, \ominus$):

$$\overline{M}^{u}_{nm} := \frac{2}{c^2} M^{u}_{nm} . \tag{31}$$

Metric perturbation due to Earth's multipoles and rotation can be written as

$$\begin{aligned}
[h^{\oplus}_{\mu\nu}] = &\sum_{nm} \overline{M}^{\oplus}_{nm} Y^m_n \times \\
&\mathrm{diag}\left(\frac{P^{(0)}_n}{r^{n+1}}, \frac{P^{(0)}_n}{r^{n-1}(r-r_s)^2}, \frac{P^{(1)}_n}{r^{n-1}}, \frac{P^{(1)}_n \sin^2\theta}{r^{n-1}} \right) \\
&+ \sum_{nm} \overline{M}^{\oplus}_{nm,T} Y^m_n \frac{P^{(3)}_n}{r^{n-1}(r-r_s)} (\delta_{\mu,1}\delta_{\nu,0} + \delta_{\nu,1}\delta_{\mu,0}) \\
&- a_{\oplus} \frac{r_s}{r} \sin^2\theta (\delta_{\mu,3}\delta_{\nu,0} + \delta_{\nu,3}\delta_{\mu,0}) ,
\end{aligned} \tag{32}$$

where Earth's multipoles $\overline{M}^{\oplus}_{nm}$ are functions of time and include rotation, ocean and solid tides; and $\overline{M}^{\oplus}_{nm,T}$ are their time derivatives.

Metric perturbations due to other celestial bodies are

$$\begin{aligned}
[h^{\ominus}_{\mu\nu}] = &\sum_{nm} \overline{M}^{\ominus}_{nm} Y^m_n \times \\
&\mathrm{diag}\left(r^n R^{(0)}_n, \frac{r^{n+2} R^{(0)}_n}{(r-r_s)^2}, r^{n+2} R^{(1)}_n, r^{n+2} R^{(1)}_n \sin^2\theta \right) \\
&+ \sum_{nm} \overline{M}^{\ominus}_{nm,T} Y^m_n \frac{r^{n+2} R^{(3)}}{r-r_s} (\delta_{\mu,1}\delta_{\nu,0} + \delta_{\nu,1}\delta_{\mu,0}) ,
\end{aligned} \tag{33}$$

where $\overline{M}^{\ominus}_{nm}$ are summed multipoles of other celestial bodies (in our case the Sun, the Moon, Jupiter and Venus).

The first order approximations of the metric perturbations given by (32) and (33) are fully determined by multipole momenta $\overline{M}^{\oplus}_{nm}$, $\overline{M}^{\ominus}_{nm}$, Kerr parameter a, and functions $P^{(i)}_n$ and $R^{(i)}_n$ (for details see [11]).

4 Conclusions

The aim of our project is to model the Galileo GNSS in general relativity including all relevant gravitational perturbations and to test to what level can this new approach improve the accuracy and stability of the Galileo GNSS reference frame.

In this contribution, we show how we can include gravitational perturbations with linear perturbation theory on a Schwarzschild background and present explicit expressions (32) and (33) for the perturbative metric.

In the next steps, we will use this metric in the perturbative Hamiltonian formalism to obtain time derivatives of zeroth order constants of motion. With the derivatives known, we will be able to determine the time evolution of these slowly changing constants of motion and apply them to analytical solutions for Schwarzschild geodesics to obtain satellites' orbits in perturbed space-time. Once the orbits are known, we can use them to do the relativistic positioning, as well as build the ABC in the perturbed space-time around Earth, using only inter-link communication between GNSS satellites.

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